

POINTWISE CONVERGENCE OF FOURIER-TYPE SERIES WITH EXPONENTIAL WEIGHTS

HEE SUN JUNG¹ AND RYOZI SAKAI²

ABSTRACT. Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \rightarrow [0, \infty)$ be an even function. We consider the exponential weights $w(x) = e^{-Q(x)}$, $x \in \mathbb{R}$. In this paper we obtain a pointwise convergence theorem for the Fourier-type series with respect to the orthonormal polynomials $\{p_n(w^2; x)\}$.

MSC: 42A20

Keywords; exponential weights, partial sum of Fourier-type series

1. INTRODUCTION AND THEOREM

Let $\mathbb{R} = (-\infty, \infty)$, and let $Q \in C^1(\mathbb{R}) : \mathbb{R} \rightarrow [0, \infty)$ be an even function. We consider the weights $w(x) := \exp(-Q(x))$. Then we suppose that $\int_0^\infty x^n w^2(x) dx < \infty$ for all $n = 0, 1, 2, \dots$

First we need the following definition from [4]. We say that $f : \mathbb{R} \rightarrow \mathbb{R}^+$ is quasi-increasing if there exists $C > 0$ such that $f(x) \leq Cf(y)$, $0 < x < y$.

Definition 1.1 (see [4]). *Let $Q : \mathbb{R} \rightarrow \mathbb{R}^+$ be a continuous even function satisfying the following properties:*

- (a) $Q'(x)$ is continuous in \mathbb{R} and $Q(0) = 0$.
- (b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
- (c) $\lim_{x \rightarrow \infty} Q(x) = \infty$.
- (d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0$$

is quasi-increasing in $(0, \infty)$ with

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R}^+ \setminus \{0\}.$$

- (e) There exists $C_1 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C_1 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus \{0\}.$$

Then we say that $w = \exp(-Q)$ is in the class $\mathcal{F}(C^2)$. Besides, if there exists a compact subinterval $J(\ni 0)$ of \mathbb{R} and $C_2 > 0$ such that

$$\frac{Q''(x)}{|Q'(x)|} \geq C_2 \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J,$$

then we say that $w = \exp(-Q)$ is in the class $\mathcal{F}(C^2+)$. If $T(x)$ is bounded, then w is called the Freud-type weight, and if $T(x)$ is unbounded, then w is the Erdős-type weight.

A typical example in $\mathcal{F}(C^2+)$ is given as follows:

Example 1.2 ([4, Example 1.2] and [1, Theorem 3.1]). For $\alpha > 1$ and a non-negative integer ℓ , we put

$$Q(x) = Q_{\ell, \alpha}(x) := \exp_{\ell}(|x|^{\alpha}) - \exp_{\ell}(0),$$

where for $\ell \geq 1$,

$$\exp_{\ell}(x) := \exp(\exp(\exp(\cdots \exp x) \cdots)) \quad (\ell\text{-times})$$

and $\exp_0(x) := x$.

We construct the orthonormal polynomials $p_n(x) = p_n(w^2, x)$ of degree n for $w^2(x)$, that is,

$$\int_{-\infty}^{\infty} p_n(x) p_m(x) w^2(x) dx = \delta_{mn} \quad (\text{Kronecker delta}).$$

Let $f w \in L_1(\mathbb{R})$. The Fourier series of f is defined by

$$\tilde{f}(x) := \sum_{k=0}^{\infty} a_k f p_k(x), \quad a_k f := \int_{-\infty}^{\infty} f(t) p_k(t) w^2(t) dt.$$

We denote the partial sum of $\tilde{f}(x)$ by

$$s_n(f, x) := s_n(w^2, f, x) := \sum_{k=0}^{n-1} a_k f p_k(x).$$

The partial sum $s_n(f, x)$ admits the representation

$$s_n(f, x) = \int_{-\infty}^{\infty} f(t) K_n(x, t) w^2(t) dt,$$

where

$$K_n(x, t) = \sum_{k=0}^{n-1} p_k(x) p_k(t).$$

Since

$$\int_{-\infty}^{\infty} K_n(x, t) w^2(t) dt = 1,$$

we have

$$(1.1) \quad s_n(f, x) - f(x) = \int_{-\infty}^{\infty} K_n(x, t) (f(t) - f(x)) w^2(t) dt.$$

The Christoffel-Darboux formula asserts that

$$(1.2) \quad K_n(x, t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_n(x) p_{n-1}(t) - p_{n-1}(x) p_n(t)}{x - t}, \quad p_n(x) =: \gamma_n x^n + \dots$$

In this paper we will show a pointwise convergence for the partial sum $s_n(f, x)$ of $\tilde{f}(x)$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function having bounded variation on every compact interval. The measure introduced by g on Borel subset of \mathbb{R} will be denoted by $|dg|$. For any interval (finite or infinite) I , we define

$$V_{\delta}(I, g) := \int_I w^{\delta}(t) |dg(t)|,$$

where $0 < \delta \leq 1$ is fixed. Let \mathcal{B}_δ denote the class of all functions g having bounded variation on \mathbb{R} , that is, $V_\delta(\mathbb{R}, g) < \infty$. We need the Mhaskar-Rakhmanov-Saff numbers a_x ;

$$x = \frac{2}{\pi} \int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du, \quad x > 0.$$

Mhaskar [5] got the following pointwise convergence theorem.

Mhaskar Theorem ([5, Theorem 9.1.2]). Let $w = \exp(-Q)$ be a Freud-type weight such that Q'' is increasing on $(0, \infty)$, $f \in \mathcal{B}_1$, and let x be a point of continuity of f . Then for $n \geq cxQ'(x)$,

$$\begin{aligned} & |s_n(f, x) - f(x)| \\ & \leq C \exp(cxQ'(x)) \left\{ \frac{1}{n} \sum_{k=1}^n V_1 \left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k} \right], f \right) + \int_{|u| \geq c_1 a_n} w(u) |df(u)| \right\}, \end{aligned}$$

where c, c_1 and C are some constants. In particular, the sequence $\{s_n(f, x)\}$ converges to $f(x)$.

We consider the Mhaskar Theorem for the Erdős-type weight $w = \exp(-Q) \in \mathcal{F}(C^2+)$.

Theorem 1.3. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$, and let $T(x)$ be unbounded. We suppose $f \in \mathcal{B}_\delta$, $0 < \delta < 1$. When x is a point of continuity of f , there exist $C > 0$, $c > 0$ and $0 < d \leq 1$ such that

$$\begin{aligned} & |s_n(f, x)| \\ & \leq C \exp(cxQ'(x)) \\ (1.3) \quad & \times \left(\frac{1}{n} \sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k} \right], f \right) + \frac{1}{n} \int_{|u| \leq a_{dn}} w^\delta(u) |df(u)| \right. \\ & \quad \left. + \frac{1}{nT^{1/4}(a_n)} \int_{|u| \leq a \frac{dn}{2}} w(u) |df(u)| + \frac{1}{T^{1/4}(a_n)} \int_{|u| \geq a \frac{dn}{2}} w(u) |df(u)| \right). \end{aligned}$$

Hence we have

$$\begin{aligned} & |s_n(f, x) - f(x)| \\ & \leq C \exp(cxQ'(x)) \\ (1.4) \quad & \times \left(\sqrt{\frac{a_n}{n}} V_\delta([x - a_n, x + a_n], f) + V_\delta \left(\left[x - \sqrt{\frac{a_n}{n}}, x + \sqrt{\frac{a_n}{n}} \right], f \right) \right. \\ & \quad + \frac{1}{n} \int_{|u| \leq a_{dn}} w^\delta(u) |df(u)| + \frac{1}{nT^{1/4}(a_n)} \int_{|u| \leq a \frac{dn}{2}} w(u) |df(u)| \\ & \quad \left. + \frac{1}{T^{1/4}(a_n)} \int_{|u| \geq a \frac{dn}{2}} w(u) |df(u)| \right). \end{aligned}$$

In particular, the sequence $\{s_n(f, x)\}$ converges to $f(x)$.

For any nonzero real valued functions $f(x)$ and $g(x)$, we write $f(x) \sim g(x)$ if there exist the constants $C_1, C_2 > 0$ independent of x such that $C_1 g(x) \leq f(x) \leq C_2 g(x)$ for all x . Similarly, for any two sequences of positive numbers $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$ we define $c_n \sim d_n$.

Throughout this paper C, C_1, C_2, \dots denote positive constants independent of n, x, t

or polynomials $P_n(x)$. The same symbol does not necessarily denote the same constant in different occurrences.

2. LEMMAS

To prove Theorem 1.3 we need some lemmas. In this paper we treat $w = \exp(-Q) \in \mathcal{F}(C^2+)$. To prove our main theorem, we use many lemmas.

Lemma 2.1. (1) [4, Lemma 3.5 (3.27)-(3.29)] *For fixed $L > 0$ and uniformly for $t > 0$,*

$$a_{Lt} \sim a_t \quad \text{and} \quad T(a_{Lt}) \sim T(a_t).$$

(2) [4, Lemma 3.4 (3.18),(3.17), Lemma 3.8 (3.42)] *For $t > 0$,*

$$Q(a_t) \sim \frac{t}{\sqrt{T(a_t)}} \quad \text{and} \quad Q'(a_t) \sim \frac{t\sqrt{T(a_t)}}{a_t}.$$

(3) [4, Lemma 3.4 (3.4)] *There exist C_1, C_2 such that for $s/r \geq 1$,*

$$\left(\frac{s}{r}\right)^{\max\{\Lambda, C_1 T(r)\}} \leq \frac{Q(s)}{Q(r)} \leq \left(\frac{s}{r}\right)^{C_2 T(r)}.$$

(4) [4, Lemma 3.11 (a),(b)] *Given fixed $0 < \alpha$, we have uniformly for $t > 0$,*

$$\left|1 - \frac{a_{\alpha t}}{a_t}\right| \sim \frac{1}{T(a_t)},$$

and there exists $C > 0$ such that for $t > 0$,

$$\left|1 - \frac{a_t}{a_{st}}\right| \geq \frac{C}{T(a_t)} \left|1 - \frac{1}{s}\right|, \quad \frac{1}{2} \leq s \leq 2.$$

In addition, for $0 < \alpha < 1$, there exists $C > 0$ such that for $s > 0$,

$$T(x) \left(1 - \frac{x}{a_s}\right) \geq C, \quad x \in [0, a_{\alpha s}].$$

(5) [4, Lemma 3.7] *For some $\varepsilon > 0$, and for large enough t ,*

$$(2.1) \quad T(a_t) \leq Ct^{2-\varepsilon}.$$

(6) [4, Theorem 3.5 (C)] *For $t \geq r > 0$ we have*

$$\frac{a_t}{a_r} \leq C \left(\frac{t}{r}\right)^{1/\Lambda}.$$

We define

$$\varphi_u(x) = \begin{cases} \frac{a_u}{u} \frac{1 - \frac{|x|}{a_{2u}}}{\sqrt{1 - \frac{|x|}{a_u} + \delta_u}}, & |x| \leq a_u; \\ \varphi_u(a_u), & a_u < |x|, \end{cases}$$

where

$$\delta_u = \{uT(a_u)\}^{-2/3}.$$

Let $0 < p < \infty$. The L_p Christoffel functions $\lambda_{n,p}(w; x)$ with a weight w are defined as follows;

$$\lambda_{n,p}(w; x) := \inf_{P \in \mathcal{P}_{n-1}} \int_{-\infty}^{\infty} |Pw|^p(u) du / |P|^p(x).$$

Then we have

$$\lambda_{n,2}(w; x) = \frac{1}{K(x, x)} = \frac{1}{\sum_{j=0}^{n-1} p_k(w^2, x)}$$

(see [4, (9.14),(9.15)]). We denote the zeros of the orthonormal polynomial $p_n(w^2, x)$ by $x_{n,n} < x_{n-1,n} < \dots < x_{1,n}$. Then we define the Christoffel numbers $\lambda_{k,n}, k = 1, 2, \dots, n$ such as $\lambda_{k,n} := \lambda_{n,2}(w, x_{k,n})$.

Lemma 2.2 ([4, Theorem 9.3 (c)]). *Let $0 < p < \infty$. Let $L > 0$. Then uniformly for $n \geq 1$ and $|x| \leq a_n(1 + L\delta_n)$, we have*

$$\lambda_{n,p}(w; x) \sim \varphi_n(x)w^p(x).$$

Lemma 2.3. (a)[4, Corollary 13.4, (12.20)] *Uniformly for $n \geq 1, 1 \leq k \leq n-1$,*

$$x_{kn} - x_{k+1,n} \sim \varphi_n(x_{k,n}) \quad \text{and} \quad 1 - \frac{x_{1n}}{a_n} \sim \delta_n.$$

Moreover,

$$\varphi_n(x_{k,n}) \sim \varphi_n(x_{k+1,n}), k = 1, 2, \dots, n-1.$$

(b) [2, Lemma 3.4 (d)] *Let $\max\{|x_{k,n}|, |x_{k+1,n}|\} \leq a_{n/2}$. Then we have for $x_{k+1,n} \leq x \leq x_{kn}$*

$$w(x_{k,n}) \sim w(x_{k+1,n}) \sim w(x).$$

So, for given $C > 0$ and $|x| \leq a_{n/3}$, if $|x - x_{k,n}| \leq C\varphi_n(x)$, then we have

$$w(x) \sim w(x_{k,n}).$$

Lemma 2.4 ([6, Lemma 3.4]). *For a certain constant $C > 0$,*

$$\frac{a_n}{n} \frac{1}{\sqrt{T(x)}} \varphi_n^{-1}(x) \leq C.$$

Lemma 2.5 ([4, Theorem 1.17, Theorem 1.18]). *Uniformly for $n \geq 1$ we have*

$$\sup_{x \in \mathbb{R}} |p_n(x)w(x)|x^2 - a_n^2|^{1/4} \sim 1,$$

and for $w(x) = \exp(-Q(x)) \in \mathcal{F}(C^2+)$,

$$\sup_{x \in \mathbb{R}} |p_n(x)w(x)| \sim a_n^{-1/2}(nT(a_n))^{1/6}.$$

Lemma 2.6 ([4, Lemma 13.9]). *Uniformly for $n \geq 1$,*

$$\frac{\gamma_{n-1}}{\gamma_n} \sim a_n.$$

Lemma 2.7. *Let $r > 1$ be fixed and $0 < p \leq \infty$. There exist $C_1, C_2 > 0$ such that for $n \geq 1$ and $P \in \mathcal{P}_m$,*

$$\|(Pw)(x)\|_{L_p(|x| \geq a_{rm})} \leq C_2 \exp\left(-C_1 \frac{m}{\sqrt{T(a_m)}}\right) \|Pw\|_{L_p(|x| \leq a_m)}.$$

Proof. From Lemma 2.1(4), we can choose a constant $0 < C < 1$ satisfying

$$a_m \left(1 + \frac{C}{T(a_m)}\right) \leq a_{rm}.$$

Putting $\tau := \frac{C}{T(a_m)}$, we see $a_m(1 + \tau) \leq a_{rm}$. By [3, Theorem 6.4] there exist $C_3, C_4 > 0$ such that for $m \geq 1, \tau \in (0, \frac{1}{T(a_m)})$ and polynomial $P \in \mathcal{P}_m$,

$$(2.2) \quad \|(Pw)(x)\|_{L_p(|x| \geq a_m(1+\tau))} \leq C_4 \exp\left(-C_3 m T(a_m) \tau^{3/2}\right) \|Pw\|_{L_p(|x| \leq a_m)}.$$

So from (2.2) we have for some $C_1 > 0$,

$$\begin{aligned} \|(Pw)(x)\|_{L_p(|x| \geq a_{rm})} &\leq \|(Pw)(x)\|_{L_p(|x| \geq a_m(1+\tau))} \\ &\leq C_4 \exp\left(-C_3 C \frac{m}{\sqrt{T(a_m)}}\right) \|(Pw)(x)\|_{L_p(|x| \leq a_m)}. \end{aligned}$$

Then we have the result putting $C_1 := C_3 C$ and $C_2 := C_4$. \square

Lemma 2.8 ([4, Theorem 10.3]). *Let $P \in \mathcal{P}_n$. When $0 < q \leq p \leq \infty$, we have for some $C > 0$,*

$$\|wP\|_{L_q(\mathbb{R})} \leq C a_n^{\frac{1}{q} - \frac{1}{p}} \|wP\|_{L_p(\mathbb{R})},$$

and when $0 < p \leq q \leq \infty$, we have for some $C > 0$,

$$\|wP\|_{L_q(\mathbb{R})} \leq C \left(\frac{n\sqrt{T(a_n)}}{a_n} \right)^{\frac{1}{p} - \frac{1}{q}} \|wP\|_{L_p(\mathbb{R})}.$$

Lemma 2.9 ([4, Theorem 1.9 infinite-finite range inequality]). *Let $0 < p \leq \infty$ and $r > 1$. Then there exist constants $C_1, C_2 > 0$ such that for some $\varepsilon > 0$, and $n > 0, P \in \mathcal{P}_n$,*

$$\|Pw\|_{L_p(a_{rn} \leq |x|)} \leq C_1 \exp(-C_2 n^\varepsilon) \|Pw\|_{L_p(|x| \leq a_n)}.$$

Lemma 2.10. *Let $p, q > 0$ and let $r > 1$. Then there exist constants $C, C_1 > 0$ such that for $P \in \mathcal{P}_{[\frac{n}{2r}]}$*

$$\left\{ \int_{|t| \geq a_{n/2}} |(Pw)(t)|^q dt \right\}^{1/q} \leq C_1 \exp\left(-\frac{C}{4r} \frac{n}{\sqrt{T(a_n)}}\right) \left\{ \int_{|t| \leq a_{n/2}} |(Pw)(t)|^p dt \right\}^{1/p}$$

Proof. Let $m := \lfloor \frac{n}{2r} \rfloor$, then we see $2rm \leq n$, and if we take n large enough, then we have $\frac{n}{4r} \leq m$. Therefore, using Lemma 2.7 and Lemma 2.8, for $P \in \mathcal{P}_m$,

$$\begin{aligned} \|(Pw)(x)\|_{L_q(a_{n/2} \leq |x|)} &\leq \|(Pw)(x)\|_{L_q(a_{rm} \leq |x|)} \\ &\leq C_1 \exp\left(-C \frac{m}{\sqrt{T(a_m)}}\right) \|(Pw)(x)\|_{L_q(|x| \leq a_m)} \\ &\leq C_1 \exp\left(-\frac{C}{4r} \frac{n}{\sqrt{T(a_n)}}\right) \begin{cases} a_m^{\frac{1}{q} - \frac{1}{p}} \|wP\|_{L_p(\mathbb{R})}, & 0 < q < p \leq \infty, \\ \left(\frac{m\sqrt{T(a_m)}}{a_m}\right)^{\frac{1}{p} - \frac{1}{q}} \|wP\|_{L_p(\mathbb{R})}, & 0 < p \leq q \leq \infty \end{cases} \\ &\leq C_1 \exp\left(-\frac{C_2}{4r} \frac{n}{\sqrt{T(a_n)}}\right) \|wP\|_{L_p(\mathbb{R})}, \end{aligned}$$

because for any fixed $\varepsilon > 0$

$$\max \left\{ a_m^{\frac{1}{q} - \frac{1}{p}}, \left(\frac{m\sqrt{T(a_m)}}{a_m} \right)^{\frac{1}{p} - \frac{1}{q}} \right\} \leq \exp\left(\frac{\varepsilon n}{\sqrt{T(a_n)}}\right).$$

Now, we may estimate $\|wP\|_{L_p(\mathbb{R})}$. Using Lemma 2.9 (infinite-finite range inequality), we have

$$\begin{aligned}\|wP\|_{L_p(\mathbb{R})} &\leq \|wP\|_{L_p(|x|\leq a_{rm})} + \|wP\|_{L_p(a_{rm}<|x|)} \\ &\leq \|wP\|_{L_p(|x|\leq a_{rm})} + C_1 \|wP\|_{L_p(|x|\leq a_m)} \\ &\leq C_2 \|wP\|_{L_p(|x|\leq a_{n/2})},\end{aligned}$$

because $rm \leq n/2$. \square

For convenience, we let $[a, b] := \{x | a \leq x \leq b\}$ if $a \leq b$, and $[a, b] := \{x | b \leq x \leq a\}$ if $b < a$.

Lemma 2.11. *Let $0 < \delta < 1$, $f \in \mathcal{B}_\delta$, $x, t \in \mathbb{R}$. Then we have for some $c_1 > 0$*

$$\begin{aligned}&w^\delta(x+t)|f(x+t) - f(x)| \\ &\leq \begin{cases} \exp(c_1 x Q'(x)) V_\delta([x, x+t], f), & \text{if } xt < 0 \text{ and } |t| < 2|x|, \\ V_\delta([x, x+t], f), & \text{otherwise.} \end{cases}\end{aligned}$$

Proof. Let $xt \geq 0$. Then

$$\begin{aligned}(2.3) \quad w^\delta(x+t)|f(x+t) - f(x)| &\leq w^\delta(x+t) \int_{[x, x+t]} |df(u)| \\ &\leq \int_{[x, x+t]} w^\delta(u) |df(u)| \leq V_\delta([x, x+t], f).\end{aligned}$$

Next, let $xt < 0$ and $|t| \geq 2|x|$. Then, for $x \leq u \leq x+t$ ($t > 0$) or $x+t \leq u \leq x$ ($t < 0$) we have $w^\delta(u) \geq w^\delta(x+t)$ because of $|u| \leq |x+t|$. So we have (2.3). Finally, we consider the case of $xt < 0$ and $|t| < 2|x|$. Let $u \in [x, x+t]$ ($t > 0$) or $u \in [x+t, x]$ ($t < 0$). If $u \leq |x+t|$, then we simply have

$$w^\delta(x+t) \leq w^\delta(u).$$

So we have the result as (2.3). Let $|x+t| < |u|$. We see that

$$|Q(u) - Q(x+t)| \leq |t| |Q'(x)| < 2|x| |Q'(x)| = 2xQ'(x).$$

Hence,

$$Q(u) - 2xQ'(x) \leq Q(x+t),$$

so

$$w^\delta(x+t) \leq \exp(2\delta x Q'(x)) w^\delta(u).$$

Therefore, as (2.3) we have the result. \square

Let

$$\chi_x(t) := \begin{cases} 1, & \text{if } t \leq x, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 2.12 ([5, Corollary 1.2.6]). *Let $x \in \mathbb{R}$ be a fixed number, and integer k be found so that $0 \leq k \leq n+1$ and $x \in (x_{k+1,n}, x_{kn}]$. Then there exist $P := P_x, R := R_x \in \mathcal{P}_{2n-1}$ such that*

$$(2.4) \quad R(t) \leq \chi_x(t) \leq P(t), \quad t \in \mathbb{R},$$

and

$$(2.5) \quad \int_{-\infty}^{\infty} |P(t) - R(t)| w^2(t) dt \leq \lambda_{k+1,n} + \lambda_{kn}.$$

Moreover, the coefficients in the polynomials P_x and R_x are measurable functions of x (in fact, they are step functions, that is, when $P_x(t)$ (or $R_x(t)$) = $\sum_{i=0}^{2n-1} a_i(x)t^i$ we have $a_i(x)$ which is constant in $[x_{k+1,n}, x_{k,n}]$, so P_x and R_x mean the polynomials $P_{(x_{k+1,n}, x_{k,n}]}$, $R_{(x_{k+1,n}, x_{k,n}]}$ defined by the interval $(x_{k+1,n}, x_{k,n}]$ which contains x).

Lemma 2.13 (cf. [5, Lemma 4.1.3]). For $x \in \mathbb{R}$ and $n = 1, 2, \dots$, we have

$$E_{1,n}(w; \chi_x) \leq C \frac{a_n}{n} w(x).$$

Proof. Using Lemma 2.12, we estimate $E_{1,n}(w^2; \chi_x)$. First, let $|x| \leq a_{n/3}$. Let k be an integer such that $x \in [x_{k+1,n}, x_{k,n}] \subset [-a_n, a_n]$. Here we note $x_{1,n} < a_n$ (see (2.3)). By Lemma 2.12 we get polynomials P and R satisfying (2.4) and (2.5), so that

$$\begin{aligned} E_{1,2n-1}(w^2; \chi_x) &\leq \int_{-\infty}^{\infty} [P(t) - \chi_x(t)] w^2(t) dt + \int_{-\infty}^{\infty} [\chi_x(t) - R(t)] w^2(t) dt \\ &\leq \lambda_{k+1,n} + \lambda_{k,n}. \end{aligned}$$

Then from Lemma 2.2 and Lemma 2.3, we have for $|x| \leq a_{n/3}$

$$\lambda_{k+1,n} + \lambda_{k,n} \leq C \varphi_n(x) w^2(x) \sim \frac{a_n}{n} \sqrt{1 - \frac{|x|}{a_n}} w^2(x) \leq C \frac{a_n}{n} w^2(x).$$

Since for the Mhaskar-Saff number $a_n(w)$ with respect to the weight $w(x) = \exp(-Q(x))$ we see $a_n(w^{1/2}) = a_{2n}(w)$, we have

$$E_{1,2n-1}(w; \chi_x) = E_{1,2n-1}((w^{1/2})^2; \chi_x) \leq C \frac{a_n(w^{1/2})}{n} w(x) = C \frac{a_{2n}(w)}{n} w(x).$$

Thus, we have the result when $|x| \leq a_{n/3}$. Next, let $|x| \geq a_{n/3}$. Then, by Lemma 2.1 (2) we have

$$Q'(x) \geq Q'(a_{n/3}) \geq C \frac{n\sqrt{T(a_{n/3})}}{2a_{n/3}} \geq C \frac{n\sqrt{T(a_n)}}{a_n}.$$

Since Q' is increasing,

$$\begin{aligned} E_{1,2n-1}(w; \chi_x) &\leq \int_{-\infty}^{\infty} (1 - \chi_x(t)) w(t) dt = \int_x^{\infty} \exp(-Q(t)) dt \\ &\leq \frac{-1}{Q'(x)} \int_x^{\infty} (-Q'(t)) \exp(-Q(t)) dt = \frac{w(x)}{Q'(x)} \\ &\leq C \frac{a_n}{n\sqrt{T(a_n)}} w(x). \end{aligned}$$

Here, we can replace $E_{1,2n-1}(w; \chi_x)$ with $E_{1,n}(w; \chi_x)$. □

Lemma 2.14. Let

$$\Lambda_n(t) := \int_t^{\infty} p_n(v) w^2(v) dv, \quad t \in \mathbb{R}.$$

Let $0 < \delta < 1$. Then there exist constants $0 < d \leq 1$ and $C > 0$ such that

$$(2.6) \quad |\Lambda_n(t)| \leq C \frac{\sqrt{a_n}}{n} w^\delta(t), \quad |t| \leq a_{dn}.$$

Proof. We consider the case of n large enough. We use $r > 1$ and $P \in \mathcal{P}_{[\frac{n}{2r}]}$ in Lemma 2.7. By Lemma 2.10, we have that for $t \in \mathbb{R}$ and $n = 1, 2, \dots$,

$$E_{1,n}(w^\delta; \chi_t) \leq C \frac{a_n/\delta}{n/\delta} w^\delta(t) \leq C \frac{a_n}{n} w^\delta(t).$$

So there exists $P \in \mathcal{P}_m$, $m = [\frac{n}{2r}]$ such that

$$(2.7) \quad \int_{\mathbb{R}} |\chi_t(u) - P(u)| w^\delta(u) du \leq C \frac{a_n}{n} w^\delta(t).$$

Here, we note that for n large enough, $\frac{1}{4}n \leq rm \leq \frac{1}{2}n$. Hence, by the orthogonal polynomial p_n and $P - 1 \in \mathcal{P}_{n-1}$, we have

$$\begin{aligned} |\Lambda_n(t)| &= \left| \int_{-\infty}^{\infty} (1 - \chi_t(u)) p_n(u) w^2(u) du \right| \\ &= \left| \int_{-\infty}^{\infty} (\chi_t(u) - P(u)) p_n(u) w^2(u) du \right| \\ &\leq \int_{|u| \leq a_{n/2}} |(\chi_t(u) - P(u)) p_n(u) w^2(u)| du \\ &\quad + \int_{|u| \geq a_{n/2}} |(\chi_t(u) - P(u)) p_n(u) w^2(u)| du \\ &=: J_1 + J_2. \end{aligned}$$

By Lemma 2.5 and (2.7) we see

$$\begin{aligned} J_1 &\leq C \frac{1}{\sqrt{a_n}} \int_{|u| \leq a_{n/2}} |(\chi_t(u) - P(u))| \left| 1 - \frac{|u|}{a_n} \right|^{-1/4} w(u) du \\ &\leq C \frac{1}{\sqrt{a_n}} \int_{|u| \leq a_{n/2}} |(\chi_t(u) - P(u)) w^\delta(u)| du \\ (2.8) \quad &\leq C \frac{\sqrt{a_n}}{n} w^\delta(t). \end{aligned}$$

Here we used the fact that $\left| 1 - \frac{|u|}{a_n} \right|^{-1/4} \leq C w^{\delta-1}(u)$ for $|u| \leq a_{n/2}$, and the Mhaskar-Rakhmanov-Saff number for the weight $w^\delta(x)$ is a_n/δ .

Next, we estimate J_2 . From (2.7), we know

$$(2.9) \quad \int_{\mathbb{R}} |P(u)| w(u) du \leq \int_{\mathbb{R}} |\chi_t(u) - P(u)| w(u) du + \int_{\mathbb{R}} |\chi_t(u)| w(u) du \leq C.$$

Since $1 - P, P \in \mathcal{P}_{[\frac{n}{2r}]}$, using Lemma 2.7, Lemma 2.8 with $p = 1$, $q = 2$ and (2.9), we have

$$\begin{aligned}
& \left\{ \int_{|u| \geq a_{n/2}} |P(u)|^2 w^2(u) du \right\}^{1/2} \leq \left\{ \int_{|u| \geq a_{r[\frac{n}{2r}]}} |P(u)|^2 w^2(u) du \right\}^{1/2} \\
& \leq C_1 \exp \left(-C \frac{n}{\sqrt{T(a_n)}} \right) \left\{ \int_{\mathbb{R}} |P(u)|^2 w^2(u) du \right\}^{1/2} \\
& \leq C_1 \exp \left(-C \frac{n}{\sqrt{T(a_n)}} \right) \left\{ \frac{n \sqrt{T(a_n)}}{a_n} \right\}^{1/2} \int_{\mathbb{R}} |(P(v))w(v)| dv \\
(2.10) & \leq C_1 \exp \left(-C_2 \frac{n}{\sqrt{T(a_n)}} \right),
\end{aligned}$$

and similarly,

$$(2.11) \quad \left\{ \int_{|u| \geq a_{n/2}} |1 - P(u)|^2 w^2(u) du \right\}^{1/2} \leq C_1 \exp \left(-C_2 \frac{n}{\sqrt{T(a_n)}} \right).$$

Since we know

$$|\chi_t(u) - P(u)|^2 \leq (|1 - P(u)|^2 + |P(u)|^2),$$

using the Schwarz inequality, we see from (2.10) and (2.11)

$$\begin{aligned}
J_2 & \leq \left(\int_{|u| \geq a_{n/2}} |\chi_t(u) - P(u)|^2 w^2(u) du \right)^{1/2} \left(\int_{-\infty}^{\infty} p_n^2(u) w^2(u) du \right)^{1/2} \\
& = \left(\int_{|u| \geq a_{n/2}} |\chi_t(u) - P(u)|^2 w^2(u) du \right)^{1/2} \\
& \leq C \left\{ \left(\int_{|u| \geq a_{n/2}} |1 - P(u)|^2 w^2(u) du \right)^{1/2} + \left(\int_{|u| \geq a_{n/2}} |P(u)|^2 w^2(u) du \right)^{1/2} \right\} \\
& \leq C_1 \exp \left(-C_2 \frac{n}{\sqrt{T(a_n)}} \right).
\end{aligned}$$

Here we will show that there exists $0 < d < 1$ such that

$$(2.12) \quad \exp \left(-\frac{C_2}{2} \frac{n}{\sqrt{T(a_n)}} \right) \leq w(t), \quad |t| \leq a_{dn}.$$

By Lemma 2.1 (3), (4), we have for some constant $0 < L < 1$

$$\frac{Q(a_{t/2})}{Q(a_t)} \leq \left(\frac{a_{t/2}}{a_t} \right)^{\max\{\Lambda, C_3 T(a_t)\}} \leq \left(1 - \frac{C_4}{T(a_t)} \right)^{\max\{\Lambda, C_3 T(a_t)\}} \leq L < 1.$$

Then for a positive integer k ,

$$\frac{Q\left(a_{\frac{t}{2^k}}\right)}{Q(a_t)} \leq L^k.$$

It means that $\frac{Q\left(\frac{a}{2^k}\right)}{Q(a_t)} \rightarrow 0$ as $k \rightarrow \infty$. Therefore, we see that for any constant $C > 0$, there exists a constant $0 < d < 1$ such that

$$(2.13) \quad \frac{Q(a_{dn})}{Q(a_n)} \leq C.$$

From Lemma 2.1 (2) and (2.13), there exist constants $C_5 > 0$ and $0 < d \leq 1$ such that

$$\frac{C_2}{2} \frac{n}{\sqrt{T(a_n)}} \geq C_5 Q(a_n) \geq Q(a_{dn}).$$

Thus, (2.12) is proved. Therefore, there exists a constant $0 < d \leq 1$ such that

$$(2.14) \quad J_2 \leq C_1 \exp\left(-\frac{C_2}{2} \frac{n}{\sqrt{T(a_n)}}\right) w(t), \quad |t| \leq a_{dn}.$$

Here, by (2.1) we see that for n large enough,

$$(2.15) \quad \exp\left(-\frac{C_2}{2} \frac{n}{\sqrt{T(a_n)}}\right) \leq C_1 \frac{1}{n} \leq C_1 \frac{\sqrt{a_n}}{n}.$$

Hence (2.14) and (2.15) imply

$$(2.16) \quad J_2 \leq C_1 \frac{\sqrt{a_n}}{n} w(t), \quad |t| \leq a_{dn}.$$

Consequently, from (2.8) and (2.16) we have the result (2.6). \square

3. PROOF OF THEOREM 1.3

Proof of Theorem 1.3. We will consider only for $x \geq 0$, because for the other cases, it can be shown similarly. Let $w = \exp(-Q) \in \mathcal{F}(C^2+)$. Let $x \in \mathbb{R}$ be fixed, then we consider $n \in \mathbb{N}$ large enough such as

$$(3.1) \quad |x| \leq a_{dn}/6,$$

where a_{dn} is defined in (2.6). Without loss of generality, we may assume that $f(x) = 0$, so, exchange $f(t) - f(x)$ with $f(t)$. Then we may estimate

$$|s_n(f, x)| = \left| \int_{-\infty}^{\infty} K_n(x, t) f(t) w^2(t) dt \right| = \left| \int_{-\infty}^{\infty} K_n(x, x+t) f(x+t) w^2(x+t) dt \right|.$$

For $|t| \geq \frac{a_n}{n}$ and a fixed x , we define

$$(3.2) \quad a_{dn}^* := a_{dn} - x.$$

Noting $f(x) = 0$, and by (1.1), we split $s_n(f, x)$ in five terms as follows:

$$s_n(f, x) = \int_{-\infty}^{\infty} K_n(x, x+t) f(x+t) w^2(x+t) dt =: \sum_{k=1}^5 I_k,$$

where, with $H(t) := K_n(x, x+t) f(x+t) w^2(x+t)$,

$$\begin{aligned} I_1 &:= \int_{|t| \leq \frac{a_n}{n}} H(t) dt, & I_2 &:= \int_{-a_{dn}^* - 2x}^{-a_n/n} H(t) dt, & I_3 &:= \int_{a_n/n}^{a_{dn}^*} H(t) dt, \\ I_4 &:= \int_{-\infty}^{-a_{dn}^* - 2x} H(t) dt, & I_5 &:= \int_{a_{dn}^*}^{\infty} H(t) dt. \end{aligned}$$

First, we estimate I_1 . Using the Schwarz inequality and the estimates on the Christoffel functions from Lemma 2.2 and Lemma 2.4 (note $\varphi_n(x) \sim n/a_n$ under the assumption (3.1)),

$$\begin{aligned} K_n^2(x, x+t) &\leq K_n(x, x)K_n(x+t, x+t) \\ &\leq C\varphi_n^{-1}(x)\varphi_n^{-1}(x+t)w^{-2}(x)w^{-2}(x+t) \\ &\leq C\left(\frac{n}{a_n}\right)^2 \sqrt{T(x+t)}w^{-2}(x)w^{-2}(x+t). \end{aligned}$$

Therefore, we have

$$(3.3) \quad H(t) \leq C \frac{n}{a_n} w^{-1}(x) f(x+t) T^{1/4}(x+t) w(x+t).$$

Hence, for some $0 < \delta < 1$ we have

$$\begin{aligned} |I_1| &\leq C \frac{n}{a_n} w^{-1}(x) \int_{|t| \leq a_n/n} |f(x+t) w^\delta(x+t)| dt \\ &\leq C \frac{n}{a_n} w^{-1}(x) \int_{|t| \leq a_n/n} w^\delta(x+t) \int_{[x, x+t]} |df(u)| dt \end{aligned}$$

(note $f(x) = 0$). By Lemma 2.3 (b) we have $w(x+t) \sim w(x)$ (note (3.1)), so

$$\begin{aligned} |I_1| &\leq C \frac{n}{a_n} w^{-1}(x) \int_{|t| \leq a_n/n} \int_{[x, x+t]} w^\delta(u) |df(u)| dt \\ &\leq C \frac{n}{a_n} w^{-1}(x) \int_{|t| \leq a_n/n} V_\delta([x, x+t], f) dt \\ (3.4) \quad &\leq C w^{-1}(x) V_\delta\left(\left[x - \frac{a_n}{n}, x + \frac{a_n}{n}\right], f\right). \end{aligned}$$

Secondly, we estimate I_3 . By (1.2) we have

$$K_n(x, x+t) = \frac{\gamma_{n-1}}{\gamma_n} \frac{p_{n-1}(x)p_n(x+t) - p_n(x)p_{n-1}(x+t)}{t}.$$

Using this, we estimate I_3 . We see that

$$I_3 := \frac{\gamma_{n-1}}{\gamma_n} \{p_{n-1}(x)I_{3,1} - p_n(x)I_{3,2}\},$$

where

$$\begin{aligned} I_{3,1} &:= \int_{a_n/n}^{a_{dn}^*} p_n(x+t) \frac{f(x+t)}{t} w^2(x+t) dt, \\ I_{3,2} &:= \int_{a_n/n}^{a_{dn}^*} p_{n-1}(x+t) \frac{f(x+t)}{t} w^2(x+t) dt. \end{aligned}$$

From $\gamma_{n-1}/\gamma_n \sim a_n$ (see Lemma 2.6), we have

$$|I_3| \leq C a_n^{1/2} w^{-1}(x) \{|I_{3,1}| + |I_{3,2}|\},$$

because we see, from Lemma 2.5, that for $|x| \leq a_{dn}/6$,

$$\max\{|p_n(x)|, |p_{n-1}(x)|\} \leq C a_n^{-1/2} w^{-1}(x) \left|1 - \frac{|x|}{a_n}\right|^{-1/4} \leq C a_n^{-1/2} w^{-1}(x).$$

We use $\Lambda_n(x)$ in Lemma 2.14. Applying integration by parts, we have

$$I_{3,1} = \frac{n}{a_n} f\left(x + \frac{a_n}{n}\right) \Lambda_n\left(x + \frac{a_n}{n}\right) - \frac{1}{a_{dn}^*} f(x + a_{dn}^*) \Lambda_n(x + a_{dn}^*) \\ - \int_{a_n/n}^{a_{dn}^*} \frac{\Lambda_n(x+t) |df(x+t)|}{t} + \int_{a_n/n}^{a_{dn}^*} \frac{\Lambda_n(x+t) f(x+t)}{t^2} dt.$$

When $0 < t \leq a_{dn}^*$, we see that $|x+t| \leq a_{dn}$ (see (3.2)). Hence, by (2.6) we have

$$\begin{aligned} \sqrt{a_n} |I_{3,1}| &\leq C \left\{ \left| f\left(x + \frac{a_n}{n}\right) \right| w^\delta\left(x + \frac{a_n}{n}\right) + \frac{1}{n} |f(a_{dn})| w^\delta(a_{dn}) \right. \\ (3.5) \quad &\left. + \frac{a_n}{n} \int_{a_n/n}^{a_{dn}^*} \frac{w^\delta(x+t) |df(x+t)|}{t} + \frac{a_n}{n} \int_{a_n/n}^{a_{dn}^*} \frac{w^\delta(x+t) |f(x+t)|}{t^2} dt \right\}. \end{aligned}$$

Since $V_\delta([x, x + \frac{a_n}{n}]) \leq V_\delta([x, x + \frac{a_n}{k}])$ for $1 \leq k \leq n$, we have from Lemma 2.11

$$\begin{aligned} w^\delta\left(x + \frac{a_n}{n}\right) \left| f\left(x + \frac{a_n}{n}\right) \right| &\leq C V_\delta\left(\left[x, x + \frac{a_n}{n}\right], f\right) \\ (3.6) \quad &\leq C \frac{1}{n} \sum_{k=1}^n V_\delta\left(\left[x, x + \frac{a_n}{k}\right], f\right) \end{aligned}$$

and

$$(3.7) \quad |f(a_{dn})| w^\delta(a_{dn}) \leq C V_\delta([x, a_{dn}], f)$$

(note $f(x) = 0$). On the other hand, substituting $u = \frac{a_n}{t}$ and decreasing of $V_\delta([x, x + \frac{a_n}{u}], f)$ for u , we have

$$\begin{aligned} \int_{a_n/n}^{a_{dn}^*} \frac{w^\delta(x+t) |f(x+t)|}{t^2} dt &\leq C \int_{a_n/n}^{a_{dn}^*} \frac{V_\delta([x, x+t], f)}{t^2} dt \\ &= \frac{1}{a_n} \int_{a_n/a_{dn}^*}^n V_\delta\left(\left[x, x + \frac{a_n}{u}\right], f\right) du \leq \frac{1}{a_n} \sum_{k=1}^n \int_k^{k+1} V_\delta\left(\left[x, x + \frac{a_n}{u}\right], f\right) du \\ (3.8) \quad &\leq \frac{1}{a_n} \sum_{k=1}^n V_\delta\left(\left[x, x + \frac{a_n}{k}\right], f\right). \end{aligned}$$

We estimate the remaining term in (3.5). Using integration by parts and (3.8), we have

$$\begin{aligned} (3.9) \quad \int_{a_n/n}^{a_{dn}^*} \frac{w^\delta(x+t) |df(x+t)|}{t} &= \frac{1}{a_{dn}^*} V_\delta([x, x + a_{dn}^*], f) - \frac{n}{a_n} V_\delta\left(\left[x, x + \frac{a_n}{n}\right], f\right) + \int_{a_n/n}^{a_{dn}^*} \frac{V_\delta([x, x+t], f)}{t^2} dt \\ &\leq C \frac{1}{a_n} \sum_{k=1}^n V_\delta\left(\left[x, x + \frac{a_n}{k}\right], f\right). \end{aligned}$$

Hence, substituting (3.6), (3.7), (3.8) and (3.9) into (3.5), we get

$$(3.10) \quad a_n^{1/2} |I_{3,1}| \leq C \frac{1}{n} \left(\sum_{k=1}^n V_\delta\left(\left[x, x + \frac{a_n}{k}\right], f\right) + V_\delta([x, a_{dn}], f) \right).$$

For $I_{3,2}$ we obtain the estimate as (3.10), so we have for a constant $\alpha > 0$,

$$(3.11) \quad |I_3| \leq Cw^{-1}(x) \frac{1}{n} \left(\sum_{k=1}^n V_\delta \left(\left[x, x + \frac{a_n}{k} \right], f \right) + V_\delta([x, a_{dn}], f) \right).$$

Thirdly, we estimate I_5 . Using (3.3), we have

$$(3.12) \quad \begin{aligned} |I_5| &\leq C \frac{n}{a_n} w^{-1}(x) \int_{a_{dn}^*}^{\infty} w(x+t) T^{1/4}(x+t) |f(x+t)| dt \\ &\leq C \frac{n}{a_n} w^{-1}(x) \{I_{5,1} + I_{5,2}\}, \end{aligned}$$

where

$$\begin{aligned} I_{5,1} &:= \int_{a_{dn}^*}^{\infty} w(x+t) T^{1/4}(x+t) \int_0^{a_{dn}^*} |df(x+u)| dt, \\ I_{5,2} &:= \int_{a_{dn}^*}^{\infty} w(x+t) T^{1/4}(x+t) \int_{a_{dn}^*}^t |df(x+u)| dt \end{aligned}$$

(note $f(x+t) = \int_0^t df(x+u)$). Since $a_{dn} - x = a_{dn}^* \leq t$, we see for $x+t = a_s \geq a_{dn}$

$$(3.13) \quad \frac{T^{1/4}(a_s)}{Q'(a_s)} \sim \frac{a_s}{sT^{1/4}(a_s)} \leq C \frac{a_n}{nT^{1/4}(a_n)},$$

because

$$\frac{a_s/s}{a_n/n} \leq C \frac{n}{s} \left(\frac{s}{n} \right)^{1/\Lambda} = \left(\frac{n}{s} \right)^{1-1/\Lambda} \leq C,$$

(see Lemma 2.1 (6)). Then for every $u \geq a_{dn}^*$,

$$(3.14) \quad \begin{aligned} \int_u^{\infty} w(x+t) T^{1/4}(x+t) dt &= \int_u^{\infty} \frac{T^{1/4}(x+t)}{Q'(x+t)} Q'(x+t) w(x+t) dt \\ &\leq C \frac{a_n}{nT^{1/4}(a_n)} w(x+u). \end{aligned}$$

Therefore, with integration by parts and (3.14),

$$(3.15) \quad \begin{aligned} |I_{5,2}| &= \left| \int_{a_{dn}^*}^{\infty} \int_u^{\infty} w(x+t) T^{1/4}(x+t) dt |df(x+u)| \right| \\ &\leq C \frac{a_n}{nT^{1/4}(a_n)} \int_{a_{dn}^*}^{\infty} w(x+u) |df(x+u)| \\ &\leq C \frac{a_n}{nT^{1/4}(a_n)} V_1([x+a_{dn}^*, \infty], f) \\ &= C \frac{a_n}{nT^{1/4}(a_n)} V_1([a_{dn}, \infty], f), \end{aligned}$$

and using (3.14) with $u = a_{dn}^*$,

$$(3.16) \quad |I_{5,1}| \leq C \frac{a_n}{nT^{1/4}(a_n)} w(x+a_{dn}^*) \int_0^{a_{dn}^*} |df(x+u)|.$$

Let

$$x + a_{\frac{dn}{2}}^* = a_{\frac{dn}{2}}.$$

Therefore, there exists $\varepsilon > 0$ such that

$$\begin{aligned}
 w(x + a_{dn}^*) \int_0^{a_{dn}^*/2} |df(x + u)| &\leq \frac{w(x + a_{dn}^*)}{w(x + a_{dn}^*/2)} \int_0^{a_{dn}^*/2} w(x + u) |df(x + u)| \\
 &\leq \frac{w(a_{dn})}{w(a_{dn}^*/2)} V_1 \left(\left[x, a_{dn}^*/2 \right], f \right) \\
 (3.17) \quad &\leq C \exp(-cn^\varepsilon) V_1 \left(\left[x, a_{dn}^*/2 \right], f \right).
 \end{aligned}$$

The last inequality holds as follows.

$$Q(a_{dn}) - Q\left(a_{dn}^*/2\right) \geq Q'(a_{dn}^*/2) \left(a_{dn} - a_{dn}^*/2\right) \geq CnT^{-1/2}(a_n) \geq Cn^\varepsilon, \quad \text{for some } \varepsilon > 0.$$

Therefore we have the last inequality in (3.17). Since we consider only n such that $|x| \leq a_{dn}/6$ when $a_{dn}^*/2 \leq u \leq a_{dn}^*$, we have $0 \leq x + u \leq x + a_{dn}^* = a_{dn}$. Moreover, there exists $c_1 > 0$ such that

$$\begin{aligned}
 w(x + a_{dn}^*) \int_{a_{dn}^*/2}^{a_{dn}^*} |df(x + u)| &\leq \int_{a_{dn}^*/2}^{a_{dn}^*} w(x + u) |df(x + u)| \\
 (3.18) \quad &\leq CV_1 \left(\left[x + a_{dn}^*/2, \infty \right], f \right) = CV_1 \left(\left[a_{dn}^*/2, \infty \right], f \right).
 \end{aligned}$$

Substituting (3.17), (3.18) into (3.16), we have

$$(3.19) \quad |I_{5,1}| \leq C \frac{a_n}{nT^{1/4}(a_n)} \left(\exp(-cn^{\varepsilon/2}) V_1 \left(\left[x, a_{dn}^*/2 \right], f \right) + V_1 \left(\left[a_{dn}^*/2, \infty \right], f \right) \right).$$

Together with (3.15), (3.19) and (3.12) we have for a constant $c_1 > 0$,

$$\begin{aligned}
 |I_5| &\leq CT^{-1/4}(a_n) w^{-1}(x) \\
 &\quad \times \left(\exp(-cn^{\varepsilon/2}) V_1 \left(\left[x, x + a_{dn}^*/2 \right], f \right) + V_1 \left(\left[a_{dn}^*/2, \infty \right], f \right) \right) \\
 (3.20) \quad &\leq CT^{-1/4}(a_n) w^{-1}(x) \left(\frac{1}{n} V_1 \left(\left[x, a_{dn}^*/2 \right], f \right) + V_1 \left(\left[a_{dn}^*/2, \infty \right], f \right) \right).
 \end{aligned}$$

Fourth, we can obtain an estimate of I_2 as I_3 . But we need to notice slightly. Let us define

$$I_2 := \frac{\gamma_{n-1}}{\gamma_n} \{p_{n-1}(x)I_{2,1} - p_n(x)I_{2,2}\},$$

where

$$\begin{aligned}
 I_{2,1} &:= \int_{-a_{dn}^*-2x}^{-a_n/n} p_n(x+t) \frac{f(x+t)}{t} w^2(x+t) dt, \\
 I_{2,2} &:= \int_{-a_{dn}^*-2x}^{-a_n/n} p_{n-1}(x+t) \frac{f(x+t)}{t} w^2(x+t) dt.
 \end{aligned}$$

Then we have

$$|I_2| \leq Ca_n^{1/2} w^{-1}(x) \{|I_{2,1}| + |I_{2,2}|\}.$$

The formula corresponding to (3.5) is

$$\begin{aligned}
 (3.21) \quad \sqrt{a_n} |I_{2,1}| &\leq C \left\{ \left| f\left(x - \frac{a_n}{n}\right) \right| w^\delta\left(x - \frac{a_n}{n}\right) + \frac{1}{n} |f(-a_{dn})| w^\delta(-a_{dn}) \right. \\
 &\quad \left. + \frac{a_n}{n} \int_{-a_{dn}^*-2x}^{-a_n/n} \frac{w^\delta(x+t) |df(x+t)|}{t} + \frac{a_n}{n} \int_{-a_{dn}^*-2x}^{-a_n/n} \frac{w^\delta(x+t) |f(x+t)|}{t^2} dt \right\}.
 \end{aligned}$$

As (3.6) we have, using Lemma 2.11,

$$(3.22) \quad w^\delta \left(x - \frac{a_n}{n} \right) \left| f \left(x - \frac{a_n}{n} \right) \right| \leq C \exp(c_1 x Q'(x)) \frac{1}{n} \sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x \right], f \right).$$

Since $a_n \geq a_{dn} \geq 2x > 0$, using Lemma 2.11 with $f(-a_{dn}) - f(x) = f(-a_{dn})$,

$$(3.23) \quad |f(-a_{dn})| w^\delta(-a_{dn}) \leq C V_\delta([-a_{dn}, x], f).$$

From Lemma 2.11 again

$$\int_{-a_{dn}^* - 2x}^{-a_n/n} \frac{w^\delta(x+t) |f(x+t)|}{t^2} dt \leq C \exp(c_1 x Q'(x)) \int_{-a_{dn}^* - 2x}^{-a_n/n} \frac{V_\delta([x+t, x], f)}{t^2} dt.$$

Here we note that in the case of $|t| \leq 2x, xt < 0$ we need the factor $\exp(c_1 x Q'(x))$. Let $u := -\frac{a_n}{t}$. Then noting the fact that $V_\delta([x - \frac{a_n}{u}, x], f)$ is a decreasing function of u , we have

$$(3.24) \quad \begin{aligned} \int_{-a_{dn}^* - x}^{-a_n/n} \frac{V_\delta([x+t, x], f)}{t^2} dt &= \frac{1}{a_n} \int_{\frac{a_n}{a_{dn}}}^n V_\delta \left(\left[x - \frac{a_n}{u}, x \right], f \right) du \\ &\leq \frac{1}{a_n} \sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x \right], f \right) \end{aligned}$$

and

$$\int_{-a_{dn}^* - 2x}^{-a_{dn}^* - x} \frac{V_\delta([x+t, x], f)}{t^2} dt \leq \frac{1}{a_{dn}} V_\delta([-a_{dn}, x], f).$$

Therefore, with (3.24) we have

$$(3.25) \quad \begin{aligned} &\int_{-a_{dn}^*}^{-a_n/n} \frac{w^\delta(x+t) |f(x+t)|}{t^2} dt \\ &\leq C \frac{\exp(c_1 x Q'(x))}{a_n} \left(\sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x \right], f \right) + V_\delta([-a_{dn}, x], f) \right). \end{aligned}$$

We estimate the remaining term in (3.21). Using integration by parts and (3.25), we have

$$(3.26) \quad \begin{aligned} &\int_{-a_{dn}^* - 2x}^{-a_n/n} \frac{w^\delta(x+t) |df(x+t)|}{t} \\ &= \frac{1}{a_{dn}^* + 2x} V_\delta([-a_{dn}, x], f) - \frac{n}{a_n} V_\delta \left(\left[x - \frac{a_n}{n}, x \right], f \right) \\ &\quad + \int_{-a_{dn}^* - 2x}^{-a_n/n} \frac{w^\delta(x+t) |f(x+t)|}{t^2} dt \\ &\leq C_1 \left(\frac{1}{a_{dn}} V_\delta([-a_{dn}, x], f) + \frac{n}{a_n} V_\delta \left(\left[x - \frac{a_n}{n}, x \right], f \right) \right) \\ &\quad + C_2 \frac{\exp(c_1 x Q'(x))}{a_n} \left(\sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x \right], f \right) + V_\delta([-a_{dn}, x], f) \right). \end{aligned}$$

Hence, substituting (3.22) (3.23), (3.25) and (3.26) into (3.21), we get
(3.27)

$$\sqrt{a_n}|I_{2,1}| \leq C \frac{\exp(c_1 x Q'(x))}{n} \left(\sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x \right], f \right) + V_\delta([-a_{dn}, x], f) \right).$$

Similarly, for $I_{2,2}$ we obtain the estimate as (3.27), so we have

$$(3.28) \quad \begin{aligned} |I_2| &\leq C w^{-1}(x) \frac{\exp(c_1 x Q'(x))}{n} \\ &\times \left(\sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x \right], f \right) + V_\delta([-a_{dn}, x], f) \right). \end{aligned}$$

Lastly, the estimate of I_4 also is obtained as I_5 . Using (3.3), we have

$$(3.29) \quad \begin{aligned} |I_4| &\leq C \frac{n}{a_n} w^{-1}(x) \int_{-\infty}^{-a_{dn}^* - 2x} w(x+t) |(T^{1/4} f)(x+t)| dt \\ &\leq C \frac{n}{a_n} w^{-1}(x) \{I_{4,1} + I_{4,2}\}, \end{aligned}$$

where

$$\begin{aligned} I_{4,1} &:= \int_{-\infty}^{-a_{dn}^* - 2x} w(x+t) T^{1/4}(x+t) \int_{-a_{dn}^* - 2x}^0 |df(x+u)| dt, \\ I_{4,2} &:= \int_{-\infty}^{-a_{dn}^* - 2x} w(x+t) T^{1/4}(x+t) \int_t^{-a_{dn}^* - 2x} |df(x+u)| dt \end{aligned}$$

(note $f(x+t) = \int_0^t df(x+u)$). Let $t \leq -a_{dn}^* - 2x$. For every $u \leq -a_{dn}$,

$$(3.30) \quad \begin{aligned} \int_{-\infty}^u w(x+t) T^{1/4}(x+t) dt &= \left| \int_{-\infty}^u w(x+t) Q'(x+t) \frac{T^{1/4}(x+t)}{Q'(x+t)} dt \right| \\ &\leq C \frac{a_n}{n T^{1/4}(a_n)} w(x+u) \end{aligned}$$

because for $x+t = -a_s \leq -a_{dn}$

$$\frac{T^{1/4}(a_s)}{Q'(a_s)} \leq C \frac{a_n}{n T^{1/4}(a_n)},$$

(see (3.13)). Therefore, with integration by parts and (3.30),

$$(3.31) \quad \begin{aligned} |I_{4,2}| &\leq C \frac{a_n}{n T^{1/4}(a_n)} \int_{-\infty}^{-a_{dn}^* - 2x} w(x+u) |df(x+u)| \\ &\leq C \frac{a_n}{n T^{1/4}(a_n)} V_1([-\infty, -a_{dn}], f). \end{aligned}$$

Just like (3.16) and (3.17), using (3.30) with $u = -a_{dn}^*$,

$$(3.32) \quad \begin{aligned} |I_{4,1}| &\leq C \frac{a_n}{n T^{1/4}(a_n)} w(-a_{dn}) \int_{-a_{dn}^* - 2x}^0 |df(x+u)| \\ &\leq C \exp(-cn^\varepsilon) V_1([-a_{dn}, x], f). \end{aligned}$$

Hence, from (3.31), (3.32) and (3.29),

$$\begin{aligned}
 |I_4| &\leq CT^{-1/4}(a_n)w^{-1}(x) \\
 &\quad \times \left(V_1([-\infty, -a_{dn}], f) + T^{1/2}(a_n) \exp(-cn^\varepsilon) V_1\left(\left[-a_{\frac{dn}{2}}, x\right], f\right) \right) \\
 (3.33) \quad &\leq CT^{-1/4}(a_n)w^{-1}(x) \left(V_1((-\infty, -a_{dn}], f) + \frac{1}{n} V_1\left(\left[-a_{\frac{dn}{2}}, x\right], f\right) \right).
 \end{aligned}$$

We summarize the above results. First, we note

$$w^{-1}(x) = \exp\left(\frac{xQ'(x)}{T(x)}\right) \leq \exp\left(\frac{1}{\Lambda}xQ'(x)\right).$$

Hence, there exists $c > 0$ such that

$$w^{-1}(x) \exp(cxQ'(x)) \leq \exp(cxQ'(x)).$$

From (3.4)

$$|I_1| \leq C \exp(cxQ'(x)) \frac{1}{n} \sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k}\right], f\right).$$

From (3.11)

$$|I_3| \leq C \exp(cxQ'(x)) \frac{1}{n} \left(\sum_{k=1}^n V_\delta\left(\left[x, x + \frac{a_n}{k}\right], f\right) + V_\delta([x, a_{dn}], f) \right).$$

From (3.20)

$$|I_5| \leq C \exp(cxQ'(x)) \left(\frac{1}{nT^{1/4}(a_n)} V_1\left(\left[x, a_{\frac{dn}{2}}\right], f\right) + \frac{1}{T^{1/4}(a_n)} V_1\left(\left[a_{\frac{dn}{2}}, \infty\right), f\right) \right).$$

From (3.28)

$$|I_2| \leq C \exp(cxQ'(x)) \frac{1}{n} \left(\sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x\right], f\right) + V_\delta([-a_{dn}, x], f) \right).$$

From (3.33)

$$|I_4| \leq C \exp(cxQ'(x)) \left(\frac{1}{nT^{1/4}(a_n)} V_1\left(\left[-a_{\frac{dn}{2}}, x\right], f\right) + \frac{1}{T^{1/4}(a_n)} V_1((-\infty, -a_{dn}], f) \right).$$

Hence, we have

$$|I_1| + |I_2| + |I_3| \leq C \exp(cxQ'(x)) \left(\frac{1}{n} \sum_{k=1}^n V_\delta\left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k}\right], f\right) + \frac{1}{n} \int_{|u| \leq a_{dn}} w^\delta(u) |df(u)| \right),$$

and

$$|I_4| + |I_5| \leq C \exp(cxQ'(x)) \left(\frac{1}{nT^{1/4}(a_n)} \int_{|u| \leq a_{\frac{dn}{2}}} w(u) |df(u)| + \frac{1}{T^{1/4}(a_n)} \int_{|u| \geq a_{\frac{dn}{2}}} w(u) |df(u)| \right).$$

Thus, we obtain (1.3). We need to show (1.4). Let $m := \lceil \sqrt{a_n n} \rceil$, then we see

$$\begin{aligned}
& \frac{1}{n} \sum_{k=1}^n V_\delta \left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k} \right], f \right) \\
&= \frac{1}{n} \sum_{k=1}^m V_\delta \left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k} \right], f \right) + \frac{1}{n} \sum_{k=m}^n V_\delta \left(\left[x - \frac{a_n}{k}, x + \frac{a_n}{k} \right], f \right) \\
&\leq \frac{m}{n} V_\delta([x - a_n, x + a_n], f) + \frac{1}{n} \sum_{k=m}^n V_\delta \left(\left[x - \frac{a_n}{m}, x + \frac{a_n}{m} \right], f \right) \\
&\leq \sqrt{\frac{a_n}{n}} V_\delta([x - a_n, x + a_n], f) + V_\delta \left(\left[x - \sqrt{\frac{a_n}{n}}, x + \sqrt{\frac{a_n}{n}} \right], f \right).
\end{aligned}$$

Therefore, the proof of (1.4) is complete. Here, it is clear that

$$\lim_{n \rightarrow \infty} \sqrt{\frac{a_n}{n}} V_\delta([x - a_n, x + a_n], f) \leq \lim_{n \rightarrow \infty} \sqrt{\frac{a_n}{n}} V_\delta(\mathbb{R}, f) = 0.$$

We have

$$V_\delta([x - \varepsilon, x + \varepsilon], f) \leq C \int_{x-\varepsilon}^{x+\varepsilon} |df(t)|,$$

and so

$$\lim_{\varepsilon \rightarrow 0} V_\delta([x - \varepsilon, x + \varepsilon], f) = 0.$$

Furthermore, we have

$$\frac{1}{n} \int_{|u| \leq a_{dn}} w^\delta(u) |df(u)| \leq \frac{1}{n} V_\delta(\mathbb{R}, f) \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

and similarly

$$\frac{1}{nT^{1/4}(a_n)} \int_{|u| \leq a_{\frac{dn}{2}}} w(u) |df(u)|, \quad \frac{1}{T^{1/4}(a_n)} \int_{|u| \geq a_{\frac{dn}{2}}} w(u) |df(u)| \rightarrow 0,$$

as $n \rightarrow \infty$. Consequently, it is proved that the sequence $\{s_n(f, x)\}$ converges to $f(x)$. \square

REFERENCES

- [1] H. S. Jung and R. Sakai, Specific examples of exponential weights, Commun. Korean Math. Soc. 24 (2009) No.2, 303-319.
- [2] H. S. Jung and R. Sakai, Mean and uniform convergence of Lagrange interpolation with the Erdős-type weights, JIA 2012, 2012:237, doi.10.1186/1029-242X-2012-237.
- [3] D. S. Lubinsky, A Survey of Weighted Polynomial Approximation with Exponential Weights, Surveys In Approximation Theory, 3(2007), 1-105 .
- [4] A. L. Levin and D. S. Lubinsky, Orthogonal Polynomials for Exponential, Weights, Springer, New York, 2001.
- [5] H. N. Mhaskar, Introduction to the Theory of Weighted Polynomial Approximation, World Scientific, Singapore, 1996.
- [6] R. Sakai and N. Suzuki, Mollification of exponential weights and its application to the Markov-Bernstein inequality, Pioneer Journal of Mathematics and Mathematical Sciences, Vol.7, no.1, pp.83-101, 2013.

¹DEPARTMENT OF MATHEMATICS EDUCATION, SUNGKYUNKWAN UNIVERSITY, SEOUL 110-745, REPUBLIC OF KOREA.

E-mail address: `hsun90@skku.edu`

²DEPARTMENT OF MATHEMATICS, MEIJO UNIVERSITY, NAGOYA 468-8502, JAPAN.
E-mail address: ryozi@crest.ocn.ne.jp